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UNIQUENESS OF EQUILIBRIUM IN PRODUCTION ECONOMIES

by

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ABSTRACT

Conditions that guarantee the uniqueness of equilibrium are crucial to applications of these models in exercises of comparative statics. Until now, most of the attention given to the uniqueness question have been focused on pure exchange economies. In this paper we use a topological index theorem to derive necessary and sufficient conditions for the uniqueness of equilibrium in economies with constant-returns production. Unfortunately, conditions that imply uniqueness appear to be too restrictive to have much applicability. We argue, for example, that the only economically interpretable restrictions that imply uniqueness are that an economy has only a single consumer or is an input-output system. Our results suggest a need for reformulation of the comparative statics method.

1. INTRODUCTION

Conditions that guarantee the uniqueness of equilibrium in models of economic competition are crucial to applications of these models in exercises of comparative statics. In such an exercise we begin by reducing the hypotheses made about an economic system to a mathematical model, usually a system of equations. Each equation involves two types of variables, endogenous variables, whose values we are interested in determining, and exogenous variables, or parameters, which describe the environment we are studying. Next, for given values of the parameters, we determine those values of the endogenous variables that solve the system of equations. Finally, and this is the step that distinguishes the comparative statics methods, we study how this solution changes as the parameters vary. The fundamental hypothesis underlying this type of analysis is that the state of the economic system can be completely specified by this solution , which is the equilibrium of the system. If, for a given vector of parameters, there is more than one solution, then the comparative statics methods breaks down. Lacking conditions that guarantee uniqueness, we must resort to considerations of historical conditions and dynamic stability, which greatly complicate the analysis.

Until now, most of the attention given to the uniqueness question has been focused on the pure exchange model, where the only economic activity is the exchange of given endowments of commodities among consumers. One reason for this bias in approach is obvious. The pure exchange model is formulated by setting the aggregate excess demands of consumers equal to zero. The excess demand functions can easily be assumed to be single-valued, continuous, and, when convenient, differentiable. In constrast, the usual formulation of the

model with constant-returns production involves upper-semi-continuous, set-valued correspondences. Although this formulation is often convenient for proving the existence of equilibrium, it is poorly suited for studying the question of uniqueness.

The cause of this difficulty lies in the nature of constant-returns production. The responses of producers to a vector of prices cannot be assumed to be single-valued. An obvious answer is to impose conditions on the production technology that ensure single-valued responses. This is easily accomplished if we are willing to accept decreasing returns to scale. In this case, the results obtained for the pure trade model can be applied directly. There are, however, two immediate drawbacks to this approach. First, the conditions obtained are often difficult to interpret and to impose. For example, some scheme must be devised to distribute profits in such an economy which would necessarily distort the income effects in changes in demand. Moreover, as Rader [29] has pointed out, such conditions as gross substitutability cannot easily be imposed on supply functions. Second, this approach lacks generality. As McKenzie [24] has argued, decreasing returns can be treated as a special case of constant returns but not the converse. A further argument is often advanced that decreasing returns cannot prevail in a production technology where all inputs are fully specified. A realistic model would undoubtedly allow for increasing returns and indivisibilities in production. The difficulties connected with increasing returns are, however, of a higher order of magnitude than those dealt with here.

In this paper we explore the question of when an economy with constant-returns production has a unique equilibrium. Our approach employs the index theorem developed by Kehoe [16, 18], who utilizes a single-valued, continuous

function whose fixed points are equivalent to equilibria of the model. Each fixed point of this function is associated with an index that is an integer determined by the local properties of this function at that point. The index theorem makes a statement about the sum of all the indices of equilibria that allows us to establish conditions sufficient for uniqueness. Furthermore, the mathematical conditions for uniqueness are necessarily, as well as sufficient, in almost all economies. The results that we derive generalize those of Dierker [7] and Varian [34], who deal with pure exchange economies.

Unfortunately, the conditions required for uniqueness of equilibrium in production economies appear to be more restrictive than those in pure exchange economies. Since the time of Wald [36], for example, it has been known that, if either the weak axiom of revealed preference or gross substitutability is satisfied by consumer excess demand function, then a pure exchange economy has a unique equilibrium. This is not the case for an economy with production. In fact, if the production technology is arbitrary, although the weak axiom is, in general, both necessary and sufficient for uniqueness, gross substitutability is neither. Since the weak axiom is an extremely restrictive assumption to impose on the aggregate excess demand function of consumers, this observation suggests that non-uniqueness of equilibrium is a less pathological situation than sometimes thought. A more subtle and far-reaching suggestion of our results is the need for a reformulation of the comparative statics method itself.

2. THE INDEX THEOREM

Let us begin by briefly summarizing the results of Kehoe [17, 18]. We specify the consumption side of an economy by an aggregate excess demand function ξ and the production side by an activity analysis matrix A. In such an economy with n goods, ξ is a function of the price vector $\pi = (\pi_1,\ldots,\pi_n)$. We assume that ξ satisfies the following assumptions:

Assumption 1 (Differentiability): $\xi: R^n_+ \backslash Z \to R^n$ is a continuously differentiable function; here Z is a subset of the boundary of R^n_+ that includes the origin. Furthermore, ξ is bounded from below R^n_+ and satisfied the condition that, if $\pi^i \to \pi$ where $\pi^i \in R^n_+ \backslash Z$ and $\pi \in Z \backslash \{0\}$, then $||\xi(\pi^i)|| \to \infty$.

Assumption 2 (Homogeneity): ξ is homogenous of degree zero; that is, $\xi(t\pi) \equiv \xi(\pi)$ for all $t > 0$.

Assumption 3 (Walras's law): ξ obeys Walras's law; that is, $\pi'\xi(\pi) \equiv 0$.

When Assumption 1 is replaced by a weakened continuity assumption, Assumptions 1-3 fully characterize aggregate excess demand functions under the general conditions of utility maximization (see McFadden, Mas-Colell, Mantel, and Richter [23]).

We further assume that the $n \times m$ activity analysis matrix A satisfies the following assumptions:

Assumption 4 (Free disposal): A includes n free disposal activities, one for each commodity; that is, the $n \times n$ matrix $-I$ is a submatrix of A.

Assumption 5 (Boundedness): There are no outputs without any inputs, that is, $\{x \in R^n | x = Ay \geq 0,\ y \geq 0\} = \{0\}$; equivalently, there exists $\bar{\pi} > 0$ such that $\bar{\pi}' A \leq 0$.

A pure exchange economy is one where the only production activities are the disposal ones, in other words, where $A = -I$.

To derive the index theorem, we define a single-valued, continuous function g whose fixed points $\hat{\pi} = g(\hat{\pi})$ are equivalent to equilibria of (ξ, A). The notion of equilibrium is defined in an obvious manner:

Definition: An equilibrium of an economy (ξ, A) is a price vector $\hat{\pi} \in R^n_+ \setminus Z$ that satisfies the following conditions:

a. $\hat{\pi}' A \leq 0$.

b. There exists $\hat{y} \geq 0$ such that $\xi(\hat{\pi}) = A\hat{y}$.

c. $\hat{\pi}' e = 1$ where $e = (1, \ldots, 1)$.

Several economists have defined single-valued functions whose fixed points are equivalent to such equilibria. They include McKenzie [24, 25], Scarf [32, pp. 162-169], and Todd [33]. The approach we employ is based on Todd's. Letting $S_A = \{\pi \in R^n \mid \pi' A \leq 0,\ \pi' e = 1\}$, we define p^{S_A} as the projection map that takes any point $q \in R^n$ into the point $p^{S_A}(q) \in S_A$ that is closest in terms of euclidean distance. Since S_A is compact and convex by Assumption 4 and definition and non-empty by Assumption 5, p^{S_A} is a continuous function. The fixed points of the function $g(\pi) = p^{S_A}(\pi + \xi(\pi))$ can easily be shown to be equivalent to equilibria of (ξ, A). The proof, which is given by Kehoe [18], is a simple application of the Kuhn-Tucker theorem. To define g on all $R^n_+ \setminus \{0\}$, we need to somehow bound ξ. Kehoe [17] presents a simple method for

doing this. Since g is a continuous mapping of the non-empty, compact, convex set S_A into itself, Brouwer's fixed point theorem implies that it has a fixed point and, hence, that (ξ,A) has an equilibrium.

To count the number of equilibria, we employ a differentiable version of the Lefschetz fixed point theorem given by Saigal and Simon [30]. There is, however, a minor problem to overcome before this theorem can be applied: The function of g is not everywhere differentiable. All we really need is that g is differentiable at its fixed points. To ensure this, we impose two additional assumptions:

Assumption 6: No column of A can be expressed as a linear combination of fewer than n other columns.

Assumption 7: Let $B(\pi)$ denote the submatrix of A whose columns are all those activities (possibly none) that earn zero profit at π. At any equilibrium $\hat{\pi}$ all activity levels $\hat{y}_b$ are strictly positive in the equation

$$\xi(\hat{\pi}) = \sum_{b\in B(\hat{\pi})} \hat{y}_b b.$$

In the following analysis we focus our attention on the derivatives of g. To simplify the presentation, we let $X \subset R^n$ be a smooth (that is, C^1) n dimensional manifold with boundary that contains S in its interior, does not contain the origin, and is compact and convex. It is a straightforward matter to extend the domain of g to all X using the approach of Saigal and Simon. We want to investigate properties of economies (ξ,A) for which 0 is a regular value of the map $(g - I): X \to R^n$; here I is the identity map on R^n.

Definition: An economy (ξ,A) that satisfies Assumption 6 and Assumption 7 and is such that $(Dg_{\hat{\pi}} - I)$ is non-singular at every equilibrium is a regular economy. Economies that are not regular are critical economies.

This definition of regular economy is equivalent to that given by Debreu [4] for the special case of a pure exchange economy with all equilibria strictly positive.

We focus our attention on regular economies because, first, almost all economies are regular and, second, regular economies possess very desirable properties. To make a statement about how common regularity is in the space of economies that satisfy Assumption 1-5, we must give this space, $\mathcal{E}$, a topological structure. We do this by defining the concept of distance between two economies (ξ^1,A^1) and (ξ^2,A^2). We endow the space of excess demand functions with the topology of uniform C^1 convergence on conpacta; in other words, we say that ξ^1 is close to ξ^2 if their values and the values of their first derivatives are close at price vectors in compact subsets of S. Recalling that the disposal activities are fixed, we endow the space of activity analysis matrices with the standard topology on $R^{n(m-n)}$; in other words, we say that A^1 is close to A^2 if their non-disposal activities are close in the sense of euclidean distance. $\mathcal{E}$ itself receives the natural product topology; (ξ^1,A^1) is close to (ξ^2,A^2) if both ξ^1 and ξ^2 are close and A^1 and A^2 are close. The meaning of the statement that almost all economies are regular is that regular economies form a subset of $\mathcal{E}$ that is open and dense in this topology.

The properties of regular economies were first studied by Debreu [4] in the context of the pure exchange model. The same properties hold for the model with constant-returns production. If (ξ,A) is regular, then it has a finite number of isolated equilibria that vary continuously with its parameters. This implies, for example, that the number of equilibria is locally constant. Furthermore, the index theorem developed by Dierker [7] for pure

exchange economies can be extended to the model with production. If we define $L_{\hat{\pi}}(g) = \text{sgn}(\det[Dg_{\hat{\pi}} - I])$ as the local Lefschetz number of a fixed point of g when $Dg_{\hat{\pi}} - I$ exists and is non-singular at every fixed point, then the Lefschetz fixed point theorem says that $\sum_{\pi=g(\pi)} L_{\pi}(g) = (-1)^n$. It will be convenient to develop expressions for the index that do not directly involve the function g. Using the chain rule, we can differentiate g to obtain

$$Dg_{\hat{\pi}} - I = (I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) - I,$$

where $C = [e \quad B(\hat{\pi})]$.

Definition. For any economy $(\xi, A) \in E$ that satisfies Assumption 6 and Assumption 7, if an equilibrium $\hat{\pi}$ is such that $[(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) - I]$, is non-singular, then index $(\hat{\pi})$ is defined as

$$(-1)^n \text{ sgn}(\det[(I - C(C'C)^{-1}C')\ (I + D\xi_{\hat{\pi}}) - I]).$$

The index theorem is an immediate consequence of the theorem of Saigal and Simon and the definitions of index$(\hat{\pi})$ and regular economy.

Theorem 1. If (ξ, A) is a regular economy, then index $(\hat{\pi})$ is defined and equal to +1 or -1 at every equilibrium and $\sum_{\pi=g(\pi)} \text{index } (\pi) = +1$.

This result is illustrated in Figure 1, where π and $g(\pi)$ may be interpreted as the first coordinates of a two commodity economy, where $\pi_2 = 1 - \pi_2$ and $g_2(\pi) = 1 - g_1(\pi)$, although we do not mean to be precise.

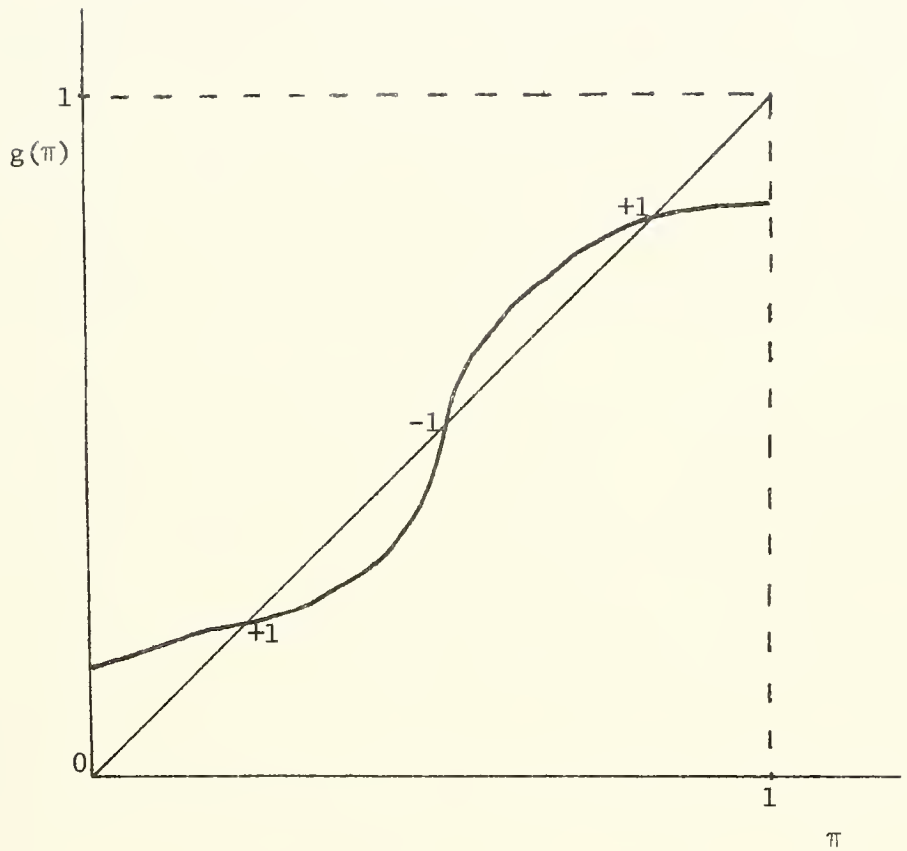

FIGURE 1

Before proceeding further, let us make two remarks about the index theorem. First, the method that we have used to derive this result is not unique. Any single-valued, continuous function whose fixed points are equivalent to equilibria and which is differentiable at its fixed points will do. It is not even necessary to employ such a function; Mas-Colell [28] has independently derived a similar result using homotopy arguments. Second, this result can easily be extended to more general production technologies. Kehoe [17] relaxes both the activity analysis specification and the free disposal assumption. The activity analysis specification can be generalized to any production technology with a finite number of production functions that possess smooth profit functions. The assumption of constant returns and Hotelling's lemma allow us to interpret the gradients of these profit functions as activities, and the calculation of the index remains the same. The free disposal assumption and be generalized to the assumption that the production cone has an interior if we are careful about redefining the domain of ξ. Letting $x \in R^n$ be some point in the interior of this cone, we calculate the index in a similar manner to that used above, replacing the vector e with -x.

Employing the index theorem presented above, we are able to count the number of equilibria of a regular economy. The formula that we have developed for this purpose, however, lacks any clear economic interpretation. To apply our result to specific economic models, we must find alternative expressions for the index of an equilibrium. One way to develop such expressions is to

manipulate the matrix $[(I - C(C'C)^{-1}C')(I + D\xi_{\hat{\pi}}) - I]$ without changing the sign of its determinant. Using elementary row and column operations with this property, we would find it a straightforward, if slightly laborious, task to demonstrate that the following formulas are equivalent to that given above:

$$\text{index } (\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \right)$$

$$= (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} D\xi_{\hat{\pi}} - E & B(\hat{\pi}) \\ B'(\hat{\pi}) & 0 \end{bmatrix} \right) .$$

Another useful formula can be derived by choosing some price, say the first, as numeraire. Let $\bar{J}$ be formed by deleting the first row and column from $D\xi_{\hat{\pi}}$ and let $\bar{B}$ be formed by deleting the first row from $B(\hat{\pi})$. If $\hat{\pi}_1 > 0$, then

$$\text{index } (\hat{\pi}) = \operatorname{sgn}\left(\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} \right) .$$

Since Assumption 7 implies that the disposal activity for commodity i is a column of $B(\hat{\pi})$ if $\hat{\pi}_i = 0$, we can also delete any row and column from $\bar{J}$, along with the row and disposal activity from $\bar{B}$, that correspond to a commodity that is a free good at equilibrium. If only one price is positive at an equilibrium, then index $(\hat{\pi}) = +1$.

3. GROSS SUBSTITUTABILITY

The most significant consequence of the index theorem is that it permits us to establish conditions sufficient for uniqueness of equilibria. If the parameters of an economy (ξ, A) are such that at every equilibrium

the index is defined and equal to +1, the set of equilibrium prices consists of a single point. A partial converse to this observation is also valid. If an economy (ξ,A) has a unique equilibrium $\hat{\pi}$, it cannot be the case that index $(\hat{\pi}) = -1$. Thus an economy with a unique equilibrium either is a critical economy or else is such that index $(\hat{\pi}) = +1$. The condition that index $(\hat{\pi}) = +1$ at every equilibrium is, therefore, necessary as well as sufficient for uniqueness in almost all cases. Consequently, it is not surprising to find that most previous theorems dealing with uniqueness of equilibrium are special cases of our uniqueness theorem.

As we have noted, the bulk of work done on the uniqueness question has dealt with the pure exchange model, which, from our perspective, is an easy case to analyze. The index theorem implies that, if $\det [- \bar{J}] > 0$ at every equilibrium of a pure exchange economy, then there is a unique equilibrium. When we restrict ourselves to economies with differentiable excess demand functions, this is the most general result possible. Indeed, any conditions that imply uniqueness must also imply that $\det [- \bar{J}] > 0$ at every equilibrium if we rule out critical economies. In any case, such conditions must imply that $\det [- \bar{J}] \geq 0$ at every equilibrium.

These conditions for uniqueness derived from the index theorem are, of course, closely related to conditions that those derived by other means.

Previous to the development of the index theorem by Dierker [7], the most general conditions for uniqueness of equilibrium were those given by Gale and Nikaido [13]. The basic result was that, if every equilibrium of a pure exchange economy is strictly positive and if det $[-\bar{J}]$ never vanishes on the domain of positive price vectors, then the economy has a unique equilibrium. Obviously, this global univalence theorem is a special case of our index theorem since, if det $[-\bar{J}]$ has the same sign at every equilibrium, then it must indeed be positive at every equilibrium. The index theorem is, however, more general than the global univalence theorem in a mathematical sense. It imposes assumptions on $[-\bar{J}]$ only at equilibria, rather than at all positive price vectors, and it is necessary as well as sufficient. Nevertheless, the easiest way to ensure that det $[-\bar{J}]$ is positive at every equilibrium is to ensure that it is positive everywhere in the domain of ξ. In practice, therefore, the mathematical conditions of the index theorem often lose their necessity when we convert them into a set of conditions on the parameters of (ξ,A) with an economic interpretation.

To illustrate the consequences of the index theorem, let us consider the case of gross substitutability.

Definition An excess demand function that satisfies Assumptions 1-3 is said to exhibit gross substitutability if $(\partial\xi_i/\partial\pi_j)(\pi) > 0$ for all $i \neq j$ and all points $\pi \in R^n_+ \backslash Z$.

It is well-known that gross substitutability in ξ implies uniqueness of equilibrium in a pure exchange economy (see Arrow, Block, and Hurwicz [1]). This same result is trivial to demonstrate using our index theorem. We first note that gross substitutability implies that any equilibrium price vector is strictly positive (Arrow and Hahn [2,pp. 221-222]). The homogeneity assumption , when differentiated, implies that $D\xi_\pi \, \pi \equiv 0$. Therefore the matrix

$$-\bar{J} = \begin{bmatrix} -\frac{\partial\xi_2}{\partial\pi_2}(\hat{\pi}) & \cdots & -\frac{\partial\xi_2}{\partial\pi_n}(\hat{\pi}) \\ \vdots & & \\ -\frac{\partial\xi_n}{\partial\pi_2}(\hat{\pi}) & \cdots & -\frac{\partial\xi_n}{\partial\pi_n}(\hat{\pi}) \end{bmatrix}$$

has positive diagonal elements and negative off-diagonal elements. Furthermore, $-\sum_{j=2}^{n} \hat{\pi}_j(\partial\xi_i/\partial\pi_j)(\hat{\pi}) = \hat{\pi}_1(\partial\xi_i/\partial\pi_1)(\hat{\pi}) > 0$, $i = 2,\ldots,n$, implies that $-\bar{J}$ has the same form as a productive Leontief matrix; that is, $-\bar{J}$ is a P matrix with all its principal minors positive. Thus, $\det[-\bar{J}] > 0$ at every equilibrium, and the index theorem implies that there is a unique equilibrium. The gross substitutability conditions can easily be weakened to $(\partial\xi_i/\partial\pi_j)(\pi) \geq 0$ for all $i \neq j$ if we make provisions to rule out critical economies.

Unfortunately, in economies with production, the gross substitutability conditions do not seem to play a significant role. If, however, $-\bar{J}$ is a positive definite matrix, a special type of P matrix, then index $(\hat{\pi})$ is positive. To demonstrate this point, we need the following lemma.

Lemma 1. Let $\bar{J}$ and $\bar{B}$ be defined as previously. If $\bar{J}$ is non-singular, then index $(\hat{\pi}) = \text{sgn}(\det[-\bar{J}] \cdot \det[-\bar{B}'\bar{J}^{-1}\bar{B}])$.

Proof. We base our argument on that of Gantmacher [14, pp. 45-46]. The expression

$$\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}$$

has the same sign as index $(\hat{\pi})$. If we pre-multiply the first row of this matrix by $\bar{B}'\bar{J}^{-1}$ and add it to the last row, the determinant stays the same. The sign of index $(\hat{\pi})$ is then equivalent to that of

$$\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ 0 & -\bar{B}'\bar{J}^{-1}\bar{B} \end{bmatrix} = \det[-\bar{J}]\cdot\det[-\bar{B}'\bar{J}^{-1}\bar{B}].$$

□

That $-\bar{J}$ is positive definite implies that $-\bar{J}^{-1}$ is positive definite, which in turn implies that $-\bar{B}'\bar{J}^{-1}\bar{B}$ is positive definite. Therefore, if $-\bar{J}$ is positive definite, index $(\hat{\pi}) = +1$. Recall, however, that although a positive definite matrix is a P matrix, the converse does not necessarily hold unless the P matrix is symmetric. There is, of course, no reason for $-\bar{J}$ to be symmetric. That $-\bar{J}$ is a P matrix does not, therefore, seem to imply anything about index $(\hat{\pi})$. Indeed, Kehoe [20] has constructed an economy with four commodities are gross substitutes that has three equilibria.

Nonetheless, it should be admitted that an example of non- uniqueness coupled with gross substitutability in demand cannot be constructed with fewer than four commodities. If $n \leq 3$, gross substitutability in demand implies uniqueness. To demonstrate this somewhat curious result, we employ the following lemma.

Lemma 2. Let $\bar{J}$ and $\bar{B}$ be defined as previously. If $\bar{B}$ has n - 1 colums, then index $(\hat{\pi}) = +1$.

Proof. If an equilibrium $\hat{\pi}$ has an $n \times (n-1)$ matrix of activities associated with it, the matrix $\bar{B}$ is square and, by Assumption 6, non-singular. Therefore

$$\det \begin{bmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix} = \det [\bar{B}'\bar{B}] .$$

Since $\bar{B}'\bar{B}$ is positive definite, index $(\hat{\pi}) = +1$. □

There are two cases to investigate, that of n = 2 and that of n = 3. If n = 2, it is obvious that any economy with gross substitutability in demand has index $(\hat{\pi}) = +1$ at every equilibrium. Either production takes place at equilibrium or it does not. If it does, there can only be 1 = n - 1 activity in use, and index $(\hat{\pi})$ is therefore +1. If it does not, index $(\hat{\pi}) = +1$ because $-\bar{J}$ is a P matrix, in fact, a positive scalar. A similar argumant works for the case n - 3. Either there are two or one or zero activities in use at equilibrium. If there are two or zero, then we can demonstrate that index $(\hat{\pi}) = +1$ using the same reasoning as above. On the other hand, if there is only one activity in use at equilibrium, the argument is slightly more involved. We can choose the commodity whose corresponding row is deleted fron the 3×1 matrix $B(\hat{\pi})$ so that both elements of $\bar{B}$ have the same sign. It follows that the 3×3 matrix

$$\begin{pmatrix} -\bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{pmatrix}$$

has one of two sign patterns

$$\begin{bmatrix} + & - & - \\ - & + & - \\ + & + & 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} + & - & + \\ - & + & + \\ - & - & 0 \end{bmatrix} .$$

In either case, the determinant is positive and index $(\hat{\pi}) = +1$.

Our arguments have produced the following theorem.

Theorem 2. Suppose an economy $(\xi, A) \in E$ is such that ξ exhibits gross substitutability. Then (ξ, A) is a regular economy and has a unique equilibrium if either of the following two additional conditions are satisfied.

a. $A = -I$; that is, (ξ, A) is a pure exchange economy.

b. $n \leq 3$ and (ξ, A) satisfies Assumption 5 and Assumption 7.

It is easy to construct an example that demonstrates that, if neither of these two conditions holds, then (ξ, A) does not necessarily have a unique equilibrium. We can certainly do away with the differentiability part of Assumption 1, as well as Assumption 6 and Assumption 7, and achieve the same result. These assumptions are necessary only because of the differentiable approach that we have chosen to take.

4. THE WEAK AXIOM OF REVEALED PREFERENCE

That ξ exhibits gross substitutability does not, in general, imply that (ξ,A) has a unique equilibrium. If, however, ξ satisfies another condition, the weak axiom of revealed preference, then (ξ,A) does have a unique equilibrium. In fact, this is the only condition that can be imposed on ξ independently of A that implies uniqueness; in other words, if ξ does not satisfy the weak axiom, then a matrix A can be constructed so that (ξ,A) has multiple equilibria.

Definition. An excess demand function ξ that satisfies Assumptions 1 - 3 is said to satisfy the weak axiom of revealed preference if, for every $\pi^1, \pi^2 \in R^n_+ \backslash Z$, $\xi(\pi^1) \neq \xi(\pi^2)$ and $\pi^{2'} \xi(\pi^1) \leq 0$ imply $\pi^{1'}\xi(\pi^2) > 0$.

The weak axiom is closely related to the negative semi-definitives of $D\xi_\pi$. We have already demonstrated that the positive definiteness of $-\bar{J}$ implies the uniqueness of equilibrium. We now mean to explore this result in more detail.

Lemma 3. Suppose that an $(\xi,A)\in E$ is such that ξ satisfies the weak axiom of revealed preference. Then, at any equilibrium $\hat{\pi}$, $D\xi_{\hat{\pi}}$ is negative semi-definite on the null space of activities in use; that is, $\pi' D\xi_{\hat{\pi}}\pi \leq 0$ for all π such that $\pi' B(\hat{\pi}) = 0$.

Proof. Much of our argument follows closely one given by Kihlstrom, Mas-Colell, and Sonnenschein [22]. Letting π^1 and $\pi^2 = \pi^1 + v$ be distinct points in $R^n_+ \backslash Z$. we define $\pi(t) = \pi^1 + tv$. Since each ξ_i is continuously differentiable and $||\pi(t) - \pi^1|| = t||v||$, we can use the definition of partial derivative to establish that

$$0 = \lim_{t\to 0} \left| \xi_i(\pi(t)) - \xi_i(\pi^1) - \sum_{j=1}^{n} (\pi_j(t) - \pi_j^1) \frac{\partial \xi_i}{\partial \pi_j}(\pi^1) \right| / t||v||, \quad i = 1,\ldots,n.$$

Since $tv_i = \pi_i(t) - \pi_i^1$, we can multiply this expression by $(||v||/t)(\pi_i(t) - \pi_i^1)$ to obtain

$$0 = \lim_{t\to 0} \left| (\pi_i(t) - \pi_i^1)(\xi_i(\pi(t)) - \xi_i(\pi^1))/t^2 - \sum_{j=1}^{n} v_i v_j \frac{\partial \xi_i}{\partial \pi_j}(\pi^1) \right|, \quad i = 1,\ldots,n.$$

Summing over i, we obtain

$$0 = \lim_{t\to 0} \left| (1/t^2)(\pi(t) - \pi^1)'(\xi(\pi(t)) - \xi(\pi^1)) - \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \frac{\partial \xi_i}{\partial \pi_j}(\pi^1) \right|.$$

Let us now suppose that $\hat{\pi}$ is an equilibrium of (ξ, A), that $\pi = \hat{\pi} + v \in R_+^n \backslash Z$ is any other point in the null space of the columns of $B(\hat{\pi})$, and that $\pi(t) = \hat{\pi} + tv$. Walras's law implies that

$$(\pi(t) - \hat{\pi})'(\xi(\pi(t)) - \xi(\hat{\pi})) = -\pi(t)'\xi(\hat{\pi}) - \hat{\pi}'\xi(\pi(t)).$$

Since $\pi'B(\hat{\pi})\hat{y} = \pi'\xi(\hat{\pi}) = 0$, $\pi(t)'\xi(\hat{\pi}) = 0$. The weak axiom therefore implies that $\hat{\pi}'\xi(\pi(t)) \geq 0$. Consequently,

$$v'D\xi_{\hat{\pi}}v = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \frac{\partial \xi_i}{\partial \pi_j}(\hat{\pi}) \leq 0.$$

Walras's law, when differentiated, implies that $\pi'D\xi_{\pi} \equiv -\xi(\pi)'$. Using this observation and the homogeneity assumption, we establish that

$$\begin{aligned}
\pi'D\xi_{\hat{\pi}}\pi &= (\hat{\pi} + v)'D\xi_{\hat{\pi}}(\hat{\pi} + v) \\
&= \hat{\pi}'D\xi_{\hat{\pi}}\hat{\pi} + v'D\xi_{\hat{\pi}}\hat{\pi} + \hat{\pi}'D\xi_{\hat{\pi}}v + v'D\xi_{\hat{\pi}}v \\
&= 0 + 0 - \xi(\hat{\pi})'v + v'D\xi_{\hat{\pi}}v \\
&= \hat{y}'B(\hat{\pi})'v + v'D\xi_{\hat{\pi}}v \\
&= v'D\xi_{\hat{\pi}}v \leq 0.
\end{aligned}$$

□

For those with intuition for the subtleties of the second order conditions of constrained optimization, the implications of this lemma for index $(\hat{\pi})$ are obvious. For others, like the writer, these implications must be spelled out.

Lemma 4. Let V be any $n \times (n - k)$ matrix whose columns span the null space of the columns of the $n \times k$ matrix $B = E(\hat{\pi})$. Then index $(\hat{\pi})$ = $\operatorname{sgn}(\det[-V'(D\xi_{\hat{\pi}} - E)V])$ where E is an n x n matrix whose every element is unity.

Proof. If (ξ, A) satisfies Assumption 6, then the $n \times n$ matrix $[V\ B]$ is non-singular at any equilibrium $\hat{\pi}$. Consequently, the determinant of the matrix

$$\begin{bmatrix} D\xi_{\hat{\pi}} - E & B \\ B' & 0 \end{bmatrix}$$

has the same sign as that of

$$\begin{bmatrix} V' & 0 \\ B' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D\xi_{\hat{\pi}} - E & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} V & B & 0 \\ 0 & 0 & I \end{bmatrix}$$

where I is $k \times k$. Consequently,

$$\text{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} V'(D\xi_{\hat{\pi}} - E)V & V'(D\xi_{\hat{\pi}} - E)B & 0 \\ B'(D\xi_{\hat{\pi}} - E)V & B'(D\xi_{\hat{\pi}} - E)B & B'B \\ 0 & B'B & 0 \end{bmatrix}\right)$$

$$= (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} V'(D\xi_{\hat{\pi}} - E)V & 0 & 0 \\ 0 & 0 & B'B \\ 0 & B'B & 0 \end{bmatrix}\right)$$

$$= \text{sgn}\,(\det[-V'(D\xi_{\hat{\pi}} - E)V]\cdot\det[B'B])$$

$$= \text{sgn}\,(\det[-V'(D\xi_{\hat{\pi}} - E)V])$$

since B'B is non-singular.

□

Combining Lemmas 3 and 4 yields the following theorem.

Theorem 3. Suppose that (ξ,A) is a regular economy. If ξ satisfies the weak axiom of revealed preferences, then it has a unique equilibrium.

Proof. If satisfies the weak axiom, then $V'\,D\xi_{\hat{\pi}}V$ is a negative semi-definite matrix. Adding the positive semi-definite matrix $-V'D\xi_{\hat{\pi}}V$ to the positive semi-definite matrix $V'EV$ produces the positive semi-definite matrix $-V'(D\xi_{\hat{\pi}} - E)V$. Since $\det[-V'(D\xi_{\hat{\pi}} - E)V] \geq 0$, index $(\hat{\pi}) = +1$ at every equilibrium as long as (ξ, A) is regular.

□

This is, of course, a well-known result, which could be proven with a far simpler argument. Assumption 6, Assumption 7 and the differentiability part of Assumption 1 be easily discarded. Again they are necessary only because we have chosen to take a differentiable approach.

Imposing the weak axiom of revealed preference on ξ implies, in general, that every principal minor of $-V'(D\xi_{\hat{\pi}} - E)V$ is positive. To ensure uniqueness, however, all we really need is that the determinant of this matrix is positive. We might, therefore, be tempted to look for

weaker conditions to impose on ξ that imply uniqueness. Unfortunately, the following theorem, which was shown to the writer by Scarf, indicates that such a search would be fruitless.

Theorem 4. Suppose that ξ is an excess demand function that satisfies Assumptions 1-3 but that violates the weak axiom of revealed preference in the sense that there exist distinct price vectors π^1, $\pi^2 \in R^n_+ \backslash Z$, one of which is strictly positive, such that $\pi^{2\prime}\xi(\pi^1) \leq 0$ and $\pi^{1\prime}\xi(\pi^2) \leq 0$.

Then there exists an activity analysis matrix A that satisfies Assumptions 4 and 5 and is such that (ξ,A) has multiple equilibria.

Proof. Let $a^1 = \xi(\pi^1)$, $a^2 = \xi(\pi^2)$, and $A = [-I\ a^1 a^2]$. This matrix obviously satisfies Assumption 4. It also satisfies Assumption 5 since either π^1 or π^2 is strictly positive and both satisfy $\pi' A \leq 0$, Since both $\pi^i / \Sigma_{j=1}^n \pi_j^i$ satisfy the equilibrium conditions, (ξ,A) has multiple equilibria. □

We require that either π^1 or π^2 is positive to ensure that A satisfies Assumption 5. If the only price vectors that violate the weak axiom lies on the same face of the boundary of R^n_+, then it may not be possible to construct a matrix A that satisfies this assumption. Thus a necessary condition for uniqueness is a slightly weaker version of the weak axiom. Notice, however, even if ξ satisfies the weak axiom, if $\xi(\pi^1) = \xi(\pi^2)$ at distinct price vectors π^1 and π^2, one of which strictly positive, then an economy (ξ, A) with non-unique equilibria can be constructed. In this case,

however, the economy must be critical since it is easy to show that the set of equilibria is convex, which implies that there is a continuum of equilibria. Thus a sufficient condition for uniqueness is a slightly stronger version of the weak axiom. These qualifications are, however, mere technicalities. Both of these conditions are equivalent to the weak axiom in almost all cases; if ξ violates the weak axiom, then it almost always does so for strictly positive price vectors, and, if ξ satisfies the weak axiom, then (ξ,A) is almost always a regular economy. In general, therefore, the weak axiom is both necessary and sufficient for uniqueness of equilibria when the activity analysis technology is arbitrary.

It is difficult, however, to give much of an economic interpretation to the condition that the aggregate excess demand function ξ satisfies the weak axiom. Although any individual excess demand function, consistent with maximization of a strictly quasi-concave utility function over a non-empty budget set, satisfies the weak axiom, this property is not preserved by aggregation. In other words, even if ξ^1 and ξ^2 satisfy the weak axiom, their sum $\xi^1 + \xi^2$ may not. It has been shown by Arrow, Block, and Hurwicz [1] that gross substitutability in ξ implies that the weak axiom holds at least in comparisons between the equilibrium of a pure exchange economy and any non-equilibrium price vector. Now gross substitutability, a fairly restrictive property, is preserved by aggregation. When production takes place at equilibrium, however, this relationship between gross substitutability and the weak axiom breaks down. It is nonetheless true that gross substitutability implies that the weak axiom holds if n is less than four. This curious result is an immediate consequence of Theorems 2

and 4 and the observation that, if an excess demand function satisfying gross substitutability violates the weak axiom, it must do so for strictly positive price vectors since Z is necessarily the entire boundary of R^n_+. It should, of course, be possible to demonstrate this result by simpler means.

5. A SINGLE CONSUMER ECONOMY

One obvious conclusion of the results of the previous section is that, if the excess demand function ξ is such that it could be derived by the maximization of utility by a single consumer, then there is a unique equilibrium. There is an interesting interpretation of this observation which relates the formula for index ($\hat{\pi}$),

$$(-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \right),$$

to the bordered Hessian of an optimization problem whose solution is an equilibrium of a single consumer economy. We specify the consumption side of such an economy using the concepts of utility function and initial endowment rather than that of excess demand. We retain the activity analysis characterization of the production side. The model presented here is intended for illustrative purposes and is not the most general possible.

We assume that the utility function $u: R^n_+ \to R$ is C^2 , strictly quasi-concave, and monotonically increasing and that the vector of initial endowments $w \in R^n$ is strictly positive. At any vector of prices $\pi \in R^n_+$ the consumer is faced with the problem of choosing the consumption bundle that maximizes utility function subject to the budget constraint:

$$\max \quad u(x)$$

$$\text{s.t.} \quad \pi' x \leq \pi' w$$

$$x \geq 0 .$$

Given our assumptions, the solution to this problem $x(\pi, \pi'w)$ varies continuously with π and $\pi'w$ over the interior of $R^n_+ \times R_+$. Furthermore, the excess demand function $\xi(\pi) \equiv x(\pi, \pi'w) - w$ is continuous and satisfies Assumptions 2 and 3, ξ is also bounded from below by $-w$ and satisfies the condition that $||\xi(\pi^i)|| \to \infty$ as $\pi^i \to \pi \in \partial R^n_+ \backslash \{0\}$; have, of course, ∂R^n_+ is the boundary of R^n_+.

Unfortunately, our assumptions are not quite sufficient to imply that ξ is everywhere differentiable. We first need to strengthen slightly our strict quasi-concavity assumption on u by putting it into differentiable form: For any $\pi \in \text{int } R^n_+$, the interior of R^n_+, we assume that the quadratic form $x'[D^2 u_{x(\pi, \pi'w)}]x < 0$ for all $x \in R^n$ satisfying $\pi'x = \pi'w$. This implies that there is no price vector π in the interior of R^n_+ such that the solution to the consumer's maximization problem is degenerate in the sense that the bordered Hessian

$$\begin{bmatrix} D^2 u_{x(\pi,\pi'w)} & \pi \\ \pi' & 0 \end{bmatrix}$$

is singular (Debreu [5]). If this condition holds, then ξ is C^2 on any open set $U \subset \text{int } R^n_+$ such that all points $x(\pi, \pi'w)$, $\pi \in U$, have the same coordinates equal to zero. There is still a technical difficulty that differentiability may fail as the demand function $x(\pi, \pi'w)$ passes through a corner colution. The excess demand function might then be kinked. It is however, easy to demonstrate that it is a generic property of the single consumer model that ξ is differentiable at every equilibrium. (Mas-Colell [27, pp. 87-90]). We therefore ignore any such non-differentiability and assume that u and w are such that ξ satisfies Assumption 1.

The indirect utility function is defined as $v(\pi,I) \equiv u(x(\pi,I))$ where $I = \pi'w$. The assumptions made on u and w imply that v is C^2 in the interior of $R^n_+ \times R_+$, strictly quasi-convex in π, monotonically decreasing in π, and monotonically increasing in I (see Diewert [8, pp. 120-124], Varian [35]). This function v is related to the excess demand function ξ by Roy's identity, $\xi_i(\pi) + w_i \equiv -\left[\partial v/\partial \pi_i\right](\pi,\pi'w) \; / \left[\partial v/\partial I\right](\pi,\pi'w)$. Using the indirect utility function, we can relate the formula for the index to the bordered Hessian of an optimization problem.

Theorem 5. The unique equilibrium of a single consumer economy (u,w,A) is equivalent to the solution to the problem

$$\begin{aligned} \min \quad & v(\pi, \pi'w) \\ \text{s.t.} \quad & \pi'A \leq 0 \\ & \pi'e = 1 \,. \end{aligned}$$

Proof. If $\hat{\pi}$ is a solution to the optimization problem, then, by the Kuhn-Tucker theorem,

$$\frac{\partial v}{\partial \pi_i}(\hat{\pi}, \hat{\pi}'w) + w_i \frac{\partial v}{\partial I}(\hat{\pi}, \hat{\pi}'w) + \sum_{j=1}^{m} a_{ij}\hat{\mu}_j + \hat{\lambda} = 0 \,, \quad i = 1, \ldots, n \,,$$

for some $\hat{\mu}_j \geq 0$, $j = 1, \ldots, m$, and $\hat{\lambda}$. Dividing each equation through by $\left[\partial v/\partial I\right](\hat{\pi}, \hat{\pi}'w) > 0$, we use Roy's identity to establish that

$$-\xi_i(\hat{\pi}) + (\sum_{j=1}^{m} a_{ij}\hat{\mu}_j + \hat{\lambda})/\frac{\partial v}{\partial I}(\hat{\pi}, \hat{\pi}'w) = 0 , \quad i = 1, \ldots, n .$$

If we define $\hat{y}_j$ as $\hat{\mu}_j/\left[\partial v/\partial I\right](\hat{\pi}, \hat{\pi}'w) \geq 0$, $j = 1, \ldots, m$, we can use Walras's law and $\hat{\pi}'A\hat{\mu} = 0$ to establish that

$$\xi(\hat{\pi}) = A\hat{y} .$$

Therefore, since $\hat{\pi}'A \leq 0$ and $\hat{\pi}'e = 1$, $\hat{\pi}$ is an equilibrium. Conversely, if $\hat{\pi}$ is an equilibrium, then it satisfies the conditions sufficient for a minimum to the optimization problem.

The strict quasi-convexity of v implies that, if a solution to the minimization problem exists, it is unique. We already know that an equilibrium exists. Therefore, the single consumer economy (u,w,A) has a unique equilibrium. □

It is the strict quasi-convexity of v that implies the uniqueness of equilibrium for (u,w,A) . The differentiable version of the quasi-convexity at a solution $\hat{\pi}$ is the second order conditions on the bordered Hessian

$$\begin{bmatrix} H & C \\ C & 0 \end{bmatrix} .$$

Here H is the matrix of second partial derivatives of $v(\hat{\pi}, \hat{\pi}'w)$ and C is the matrix of coefficients of those constraints that hold with equality at $\hat{\pi}$. Once again, $C = [e \;\; B(\hat{\pi})]$. If $B(\hat{\pi})$ is $n \times k$,

then second order conditions require that

$$(-1)^{k+1}\det\begin{bmatrix} H & C \\ C & 0 \end{bmatrix} \geq 0.$$

The first derivative of v with respect to π_j is $[\partial v/\partial \pi_i](\hat{\pi}, \hat{\pi}'w) + w_i[\partial v/\partial I](\hat{\pi}, \hat{\pi}'w) \equiv -\xi_i(\hat{\pi})[\partial v/\partial I](\hat{\pi}, \hat{\pi}'w)$. Therefore the typical entry in H can be expressed as

$$h_{ij} = -\frac{\partial v}{\partial I}\frac{\partial \xi_i}{\partial \pi_j} - \xi_i(\hat{\pi})\left[\frac{\partial^2 v}{\partial I \partial \pi_j} + w_j \frac{\partial^2 v}{\partial I^2}\right].$$

Here all partial derivatives are evaluated at $(\hat{\pi}, \hat{\pi}'w)$ and $\hat{\pi}$. Now the quasi-convexity of v implies that

$$(-1)^{k+1}\det\begin{bmatrix} h_{11} \cdots h_{1n} & 1 & b_{11} \cdots b_{1k} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ h_{n1} \cdots h_{nn} & 1 & b_{n1} \cdots b_{nk} \\ 1 \cdots 1 & 0 & 0 \cdots 0 \\ b_{11} \cdots b_{n1} & 0 & 0 \cdots 0 \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ b_{1k} \cdots b_{nk} & 0 & 0 \cdots 0 \end{bmatrix} \geq 0$$

We can multiply each of the final k columns of this matrix by $\hat{y}_b([\partial^2 v/\partial I \partial \pi_j] + w_j[\partial^2 v/\partial I^2])$ and add it to the j^{th} column, $j = 1, \ldots, n$, without changing the sign of its determinant. The equilibrium condition $\xi(\hat{\pi}) = \sum_{b \in B(\hat{\pi})} \hat{y}_b b$ implies that

$$(-1)^{k+1}\det\begin{bmatrix} -\frac{\partial v}{\partial I}\frac{\partial \xi_1}{\partial \pi_1} & \cdots & -\frac{\partial v}{\partial I}\frac{\partial \xi_1}{\partial \pi_n} & 1 & b_{11}\cdots b_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ -\frac{\partial v}{\partial I}\frac{\partial \xi_n}{\partial \pi_1} & \cdots & -\frac{\partial v}{\partial I}\frac{\partial \xi_n}{\partial \pi_n} & 1 & b_{n1}\cdots b_{nk} \\ 1 & \cdots & 1 & 0 & 0 \cdots 0 \\ b_{11} & \cdots & b_{n1} & 0 & 0 \cdots 0 \\ \vdots & & \vdots & \vdots & \vdots \\ b_{1k} & & b_{nk} & 0 & 0 \quad 0 \end{bmatrix} \geq 0$$

Factoring out $dv/dI > 0$, we are left with

$$(-1)^{k+1}\left(\frac{\partial v}{\partial I}\right)^{n-k-1}\det\begin{bmatrix} -D\xi_{\hat{\pi}} & e & B(\hat{\pi}) \\ e' & 0 & 0 \\ B'(\hat{\pi}) & 0 & 0 \end{bmatrix} \geq 0.$$

This condition can be rewritten as

$$(-1)^{n}\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \geq 0.$$

Thus, when the economy (ξ,A) is regular, the second order conditions for our optimization problem imply that index $(\hat{\pi}) = +1$ at every equilibrium. In this case the index theorem gives us a restatement of what we already know: A single consumer economy has a unique equilibrium.

The restriction that ξ behaves like the excess demand of a single consumer is obviously stringent. There are two different sets of conditions that imply that this restriction holds. Both require that all demands are homogeneous of degree one in $\pi'w$; that is, $x^j(\pi, t\pi'w) \equiv tx^j(\pi, \pi'w)$ for all $t > 0$ and all consumers j. If, in addition, either all utility functions are identical (Gorman [15]) or all initial endowments are proportional (Chipman [3]), then the aggregate excess demand function behaves like that of a single consumer.

We should note, however, that the single consumer assumption is not necessary to ensure that the weak axiom of revealed preference is satisfied. Utility maximization by a single consumer is equivalent to a stronger restriction on ξ, the strong axiom of revealed preference. An example due to Gale [11] demonstrates that the strong axiom and the weak axiom are not equivalent. Unfortunately, there does not appear to be an obvious economic interpretation for excess demand functions that satisfy the weak axiom but not the strong axiom. Our goal is to find conditions for uniqueness that can be checked for by examining the parameters of (ξ, A). Searching for such conditions on the demand side of the economy alone has produced only the extremely restrictive single consumer assumption.

6. THE NON-SUBSTITUTION THEOREM

We now turn our search for conditions that ensure uniqueness of equilibrium to the production side of (ξ,A). If a search on the demand side alone produces little, a search on the production side alone does discouragingly worse. Indeed, with the exception of the restriction that there is only one positive price vector π for which $\pi'A \leq 0$, no condition on A implies that (ξ,A) has a unique equilibrium if the only restrictions placed on ξ are Assumptions 1-3. For example, $\xi(\pi) \equiv 0$ produces an infinite number of equilibria as long as there is more than one π for which $\pi'A \leq 0$. Although the economy in this instance is critical, the following theorem demonstrates that non-uniqueness is by no means pathological in these circumstances.

Theorem 6. Suppose that the activity analysis matrix A satisfies Assumptions 4-6. Then there exists an open set of excess demand functions ξ such that the economies (ξ,A) have multiple equilibria

Proof. Combining Assumptions 4-6 yields the conclusion that there exists $\bar{\pi} \in S = \{\pi \in R^n \mid e'\pi=1, \pi \geq 0\}$ strictly positive and such the $\bar{\pi}'A < 0$. If n is even, we define the function ξ^0: int $R^n_+ \to R^n$ by the rules

$$\xi^0_i(\pi) = \bar{\pi}_i - \frac{\pi_1 \bar{\pi}_i^2}{\pi_i \bar{\pi}_1}, \quad i = 1,\ldots,n,$$

$$\xi^0_1(\pi) = \frac{1}{\pi_1} \Sigma^n_{i=2} \pi_i \xi^0_i(\pi)$$

Notice the $\xi^0(\bar{\pi}) = 0$ and $-\bar{J} = -I$ when evaluated at $\bar{\pi}$. Consequently,

$$\text{index}(\bar{\pi}) = \text{sgn}(\det[-I]) = (-1)^{n-1} = -1 .$$

If n is odd, we can easily change $\xi_2^0(\pi)$ to $(\pi_1\bar{\pi}_2^2)/(\pi_2\bar{\pi}_1)-\bar{\pi}_2$ and acheive the same result.

In either case, ξ is C^1 and satisfies Assumptions 2 and 3. It does not satisfy Assumption 1, however, because it is not bounded from below on int R_+^n. This problem is easily remedied by combining ξ^0 with any excess demand function ξ^1 that satisfies Assumptions 1-3. We choose two open neighborhoods of $\bar{\pi}$, U and V, such that the closure of U, cl(U), contained in V and cl(V) $\subset$ int R_+^n. We also choose $\theta: S \to R$ as a C^1 function that satisfies

$$\theta(\pi) = 0 \qquad \text{if } \pi \in S \setminus V$$

$$\theta(\pi) = 1 \qquad \text{if } \pi \in \text{cl}(U) \cap S$$

$$0 < \theta(\pi) < 1 \qquad \text{if } \pi \in (V \setminus \text{cl}(U)) \cap S.$$

Finally, we define $\xi: R_+^n \setminus \{0\} \to R^n$ by the rule

$$\xi(\pi) = \theta\left(\frac{\pi}{e'\pi}\right)\xi^0(\pi) + \left(1-\theta\left(\frac{\pi}{e'\pi}\right)\right)\xi^1(\pi).$$

Now ξ satisfies Assumptions 1-3 and is such that (ξ,A) has $\bar{\pi}$ as an equilibrium with $\text{index}(\bar{\pi}) = -1$. (ξ, A) must therefore have multiple equilibria. Although (ξ,A) itself may not be a regular economy because of problems at other equilibria, the density of regular economies implies that we can perturb ξ slightly to produce a regular economy with multiple equilibria. The openness of regular economies now implies our contention.

□

This result indicates that to find conditions that guarantee the uniqueness of equilibrium, we must somehow combine conditions from the consumption side with ones from the production side. Lemma 2 implies that, if (ξ,A) satisfies Assumption 6 and has n-1 activities at use at every equilibrium, then it is a regular economy with a unique equilibrium. To gain some insight into this observation, recall that the conditions of the well-known non-substitution theorem of input output analysis imply that there are always n-1 activities in use at equilibrium (see Samuelson [31]).

Definition. An input-output economy is an economy (ξ,A) that satisfies the following conditions:

a. There is one non-produced good; $a_{nj} \leq 0$ for $j=1,\ldots, m$.

b. $\xi_i(\hat{\pi}) > 0$, $i=1,\ldots,n-1$, at every equilibrium $\hat{\pi}$.

c. There is no joint production; $a_{ij} > 0$ for at most one i, $j=1,\ldots m$.

d. There exists some vector of non-negative activity levels $\hat{y}$ such that $\Sigma_{j=1}^{m}, a_{ij}\hat{y}_j > 0$, $i=1,\ldots,n-1$.

One way to ensure that condition holds is to allow initial endowments only of the non-produced good, traditionally labor. Since it must be the case that all n-1 non-produced goods are produced at equilibrium, and since we rule out joint production, n-1 activities must be run at positive levels at every equilibrium $\hat{\pi}$. This implies that (ξ,A) satisfies Assumption 7 if it satisfies Assumption 6 because n-1 is the maximum number of activities that can earn zero profit at $\hat{\pi}$. We have therefore derived the following theorem with little effort.

Theorem 7. Suppose (ξ,A) is an input-output economy that satisfies Assumption 6. Then (ξ,A) is regular and has a unique equilibrium.

The non-substitution theorem implies that the unique equilibrium of (ξ,A) is determined independently of ξ, a conclusion that our theorem misses. To offset this shortcoming, we should note that Lemma 2 is a very general mathematical result. All we need to know is that the maximum possible number of activities are at use at every equilibrium to guarantee uniqueness. Unfortunately, the conditions of an input-output economy seem to be the only conditions on the parameters of (ξ,A) that ensure this premise holds.

7. AN INPUT-OUTPUT MODEL WITH TWO FACTORS

Our search for conditions that guarantee uniqueness of equilibrium has so far produced discouragingly little. Let us now experiment with a different approach. If we cannot find general conditions to impose on the parameters of (ξ, A), perhaps we can develop a method that finds all of the equilibria of (ξ, A). We then would know whether (ξ, A) has a unique equilibrium. Scarf [32] first developed an algorithm that computes an equilibrium of (ξ, A). Unfortunately, the equilibria that it computes always have positive index (see Eaves and Scarf [9]). No method is now known that finds all the equilibria of an economy that does not involve an exhaustive search.

For an economy with two commodities such a search is often possible, and indeed can be performed graphically. We can use this observation to explore the question of uniqueness of equilibrium in a class of economies that generalize the input-output concept used in the previous section.

Definition. An input-output economy with two factors is an economy (ξ, A) that satisfies the following conditions:

a. There are two non-produced goods; $a_{ij} \leq 0$ for $i = n-1$ and n, $j = 1, \ldots, m$.

b. $\xi_i(\hat{\pi}) > 0$, $i = 1, \ldots, n-2$, for every equilibrium $\hat{\pi}$.

c. There is no joint production; $a_{ij} > 0$ for at most one i, $j = 1, \ldots m$.

d. There exists some non-negative vector of activity levels $\hat{y}$ such that $\Sigma_{j=1}^{m} a_{ij} \hat{y}_i > 0$, $i = 1, \ldots n-2$.

To simplify matters we assume that (ξ, A) is a regular economy for the rest of this section.

We denote the prices of the first n-2 goods by the vector p and the prices of the two non-produced goods by the vector q. Thus, any $n\times 1$ vector of prices π can be expressed as (p,q). In addition, we partition the production matrix A into $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where A_1 is $(n-2)\times m$ and A_2 is $2 \times m$ and the excess demand vector $\xi(\pi)$ into $\begin{bmatrix} \xi^1(\pi) \\ \xi^2(\pi) \end{bmatrix}$ where $\xi^1(\pi)$ is $(n-2) \times 1$ and $\xi^2(\pi)$ is 2×1.

We begin with the case where A consists of only 2n-2 activities, the n disposal activities and an $n\times(n-2)$ matrix B with one activity to produce each of the first n-2 goods. We partition B into $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ where B_1 is $(n-2) \times (n-2)$ and B_2 is $2 \times (n-2)$. Our assumptions on (ξ, A) imply that B_1 is a productive Leontief matrix. The non-degeneracy Assumption 6 ensures that B_1 is also indecomposable. Consequently, B_1^{-1} has all elements strictly positive (see Debreu and Herstein [6]).

Let us observe that at any equilibrium $\hat{\pi} = (\hat{p},\hat{q})$ the condition $B_1\hat{y} = \xi^1(\hat{p},\hat{q})$ implies that $\hat{y} = B_1^{-1}\xi^1(\hat{p},\hat{q}) > 0$. Let us further observe that the zero profit condition $\hat{p}'B_1 + \hat{q}'B_2 = 0$ implies that $\hat{p} = -(B_2B_1^{-1})'\hat{q}$.

Since all the coordinates of $(\hat{p},\hat{q})$ cannot be equal to zero, both the coordinates of $\hat{q}$ cannot be equal to zero. Moreover, as a consequence of Assumption 6 the $(n-2)\times 2$ matrix $-(B_2B_1^{-1})'$ must have all elements strictly positive. Therefore $\hat{p}$ is strictly positive. We can use these observations to construct a pure exchange model in the space of the two factors whose equilibria are equivalent to those of (ξ,A).

We define the 2×1 excess demand function

$$z(q) = \xi^2(-(B_2B_1^{-1})'q,\ q) - B_2B_1^{-1}\xi^1(-(B_2B_1^{-1})'q,\ q) .$$

Note that z is C^1 over int R^2_+ since ξ is C^1 over int R^n_+. Therefore z satisfies a two dimensional version of Assumption 1. It also satisfies Assumptions 2 and 3 since ξ does. We observe the $\hat{q}$ is obviously an equilibrium of the two dimensional economy $(z, -1)$ if $(\hat{p},\hat{q})$ is an equilibrium of the n dimensional economy (ξ, A). (Of course, we have to allow for the possibility of disposal of one of the factors taking place at equilibrium and also normalize in both cases so that prices sum to unity.) It is also easy to demonstrate the converse: $(\hat{p},\hat{q}) = (-(B_2B_1^{-1})'\hat{q},\ \hat{q})$ is an equilibrium of (ξ, A) if $\hat{q}$ is an equilibrium of $(z, -I)$.

Since the equilibria of (ξ, A) and $(z, -I)$ are, in some sense, equivalent, we should expect that the index of an equilibrium $\hat{q}$ of $(z, -I)$ is the same as that of the corresponding equilibrium $(\hat{p},\hat{q})$ of (ξ, A). Let us first consider the case where both coordinates of $\hat{q}$ are strictly positive. The index of the equilibrium $(\hat{p},\hat{q})$ of the n dimensional economy (ξ, A) can be expressed as

$$\text{index } (\hat{p},\hat{q}) = (-1)^n \text{sgn}\left[\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi^1_{\hat{p}} & D\xi^1_{\hat{q}} & B_1 \\ e & D\xi^2_{\hat{p}} & D\xi^2_{\hat{q}} & B_2 \\ 0 & B_1' & B_2' & 0 \end{bmatrix}\right]$$

where each of the vectors e is of the appropriate dimension. If we subtract the second row of this matrix, pre-multiplied by $B_2B_1^{-1}$, from the third, we do not change its determinant. Thus, index $(\hat{p},\hat{q})$ has the same sign as

$$(-1)^n\det\begin{bmatrix} 0 & e' & e' & 0 \\ e & D\xi^1_{\hat{p}} & D\xi^1_{\hat{q}} & B_1 \\ e - B_2B_1^{-1}e & D\xi^2_{\hat{p}} - B_2B_1^{-1}D\xi^1_{\hat{p}} & D\xi^2_{\hat{q}} - B_2B_1^{-1}D\xi^1_{\hat{q}} & 0 \\ 0 & B_1' & B_2' & 0 \end{bmatrix},$$

which equals

$$(-1)^n\det\begin{bmatrix} 0 & e' & e' & 0 \\ 0 & 0 & 0 & B_1 \\ e - B_2B_1^{-1}e & D\xi^2_{\hat{p}} - B_2B_1^{-1}D\xi^1_{\hat{p}} & D\xi^2_{\hat{q}} - B_2B_1^{-1}D\xi^1_{\hat{q}} & 0 \\ 0 & B_1' & B_2' & 0 \end{bmatrix}.$$

Repeating this same procedure, subtracting the second column of this matrix post-multiplied by $(B_2B_1^{-1})'$ from the third, we are left with

$$(-1)^n\det[-B_1'B_1]\det\begin{bmatrix} 0 & e' - e'(B_2B_1^{-1})' \\ e - B_2B_1^{-1}e & D\xi^2_{\hat{q}} - B_2B_1^{-1}D\xi^1_{\hat{q}} - D\xi^2_{\hat{p}}(B_2B_1^{-1})' \\ & + B_2B_1^{-1}D\xi^1_{\hat{p}}(B_2B_1^{-1})' \end{bmatrix},$$

which has the same sign as

$$(-1)^2 \det \begin{bmatrix} 0 & e' - e'(B_2B_1^{-1})' \\ e - B_2B_1^{-1}e & D\xi^2_{\hat{q}} - B_2B_1^{-1}D\xi^1_{\hat{q}} - D\xi^2_{\hat{p}}(B_2B_1^{-1})' + B_2B_1^{-1}D\xi^1_{\hat{p}}(B_2B_1^{-1})' \end{bmatrix}$$

Observe, however, that

$$Dz_{\hat{q}} = D\xi^2_{\hat{q}} - B_2B_1^{-1}D\xi^1_{\hat{q}} - D\xi^2_{\hat{p}}(B_2B_1^{-1})' + B_2B_1^{-1}D\xi^1_{\hat{p}}(B_2B_1^{-1})' .$$

Consequently,

$$\text{index } (\hat{p},\hat{q}) = \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e - e'(B_2B_1^{-1})' \\ e - B_2B_1^{-1}e & Dz_{\hat{q}} \end{bmatrix}\right) .$$

When we rescale $\hat{q}$ and $Dz_{\hat{q}}$ so that $\hat{q}'e = 1$, this becomes the formula for index $(\hat{q})$ of the two dimensional economy $(z, -I)$. (Recall that in the n dimensional case $\hat{q}'(e - B_2B_1^{-1}e) = \hat{q}'e + \hat{p}'e = 1$ while in the two dimensional case $\hat{q}'e = 1$.)

If neither $\hat{q}_1$ nor $\hat{q}_2$ is equal to zero, then the index of equilibrium $\hat{q}$ can be calculated as $\operatorname{sgn}(-[\partial z_1/\partial q_1](\hat{q}))$. Alternatively, we can use the homogeneity assumption to establish that index $(\hat{q}) = \operatorname{sgn}(-[\partial z_1/\partial q_1](\hat{q}) + [\partial z_1/\partial q_2](\hat{q}))$. Walras's law implies that, if $z_1(\hat{q}) = 0$, then $z_2(\hat{q}) = 0$. Therefore equilibria of $(z, -I)$, with the possible exceptions of $(1,0)$ and $(0,1)$, are equivalent to zeros of the excess demand function $z_1(q_1, 1 - q_1)$. The index of an equilibrium $\hat{q}$ is equal to $+1$ if $z_1(q_1, 1 - q_1)$ crosses the axis from above at $\hat{q}$ and equal to -1 if it crosses from below. If either $(1,0)$ or $(0,1)$ is an equilibrium, then the index is equal to $+1$ by Lemma 2.

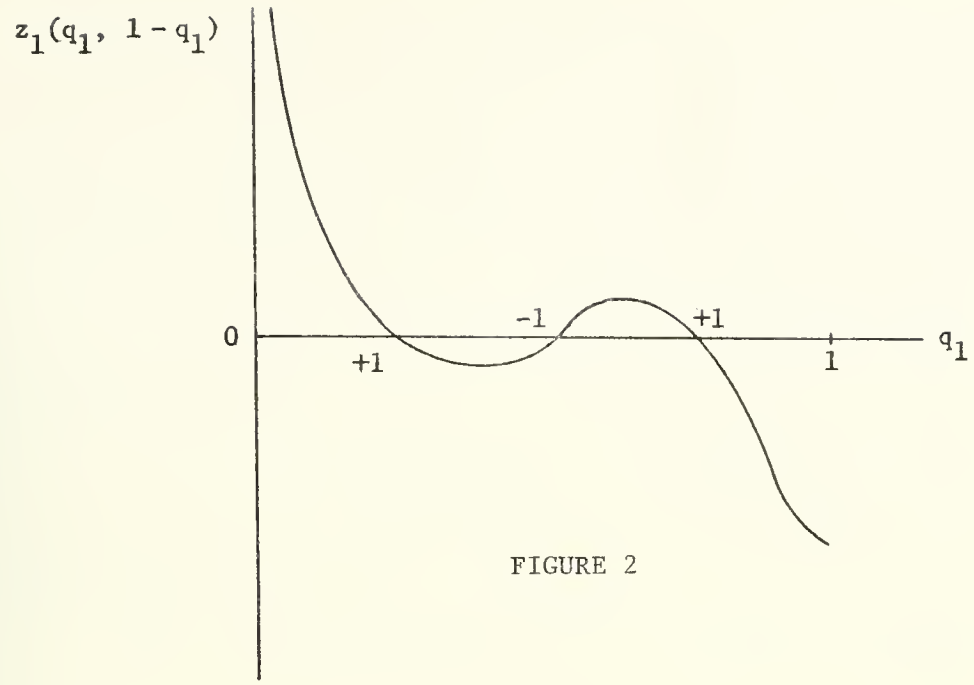

FIGURE 2

When the matrix A contains more than $2n-2$ columns, the situation is not so simple. The natural definition for $z(q)$ is that of an upper-semi-continuous, set-valued correspondence rather than that of a differentiable function. We construct the excess demand correspondence $z(q)$ so that it satisfies Assumptions 2 and 3. If we extend the definition of an economy so that excess demand can have multiple values at certain prices, the equilibria of $(z, -I)$ are again equivalent to those of (ξ, A).

For a fixed vector of factor prices q we begin by solving the linear programming problem

$$\begin{aligned} \min \quad & -q'A_2y \\ \text{s.t.} \quad & A_1y \geq e \\ & y \geq 0 . \end{aligned}$$

We have already assumed that the constraints of the problem are feasible.

Assumption 5 assures us that this problem has a finite solution. The solution is associated with a feasible basis $\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix}$ and a vector of commodity prices p such that $p'B_1 + q'B_2 = 0$ and $p'A_1 + q'A_2 \leq 0$ (see Gale [12, pp. 301-306]). When the basis $\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix}$ is uniquely defined, we set

$$z(q) = \xi^2(-(B_2B_1^{-1})'q,\ q) - B_2B_1^{-1}\xi^1(-(B_2B_1^{-1})'q,\ q)$$

and proceed as before.

A problem arises, however, when the feasible basis associated with the solution to the linear programming problem is not uniquely defined. There can be $n-1$ activities earning zero profit at prices $(-(B_2B_1^{-1})'q,\ q)$. This is the case when there are two distinct bases, $\begin{bmatrix} -q'C_2 \\ C_1 \end{bmatrix}$ and $\begin{bmatrix} -q'B_2 \\ B_1 \end{bmatrix}$, possible in the solution. Our non-degeneracy Assumption 6 implies that there can be at most two distinct bases and they differ by only one activity. At any factor prices q where the linear programming problem is degenerate in this way the final demand for the factors can be either

$$z^B(q) = \xi^2(-(B_2B_1^{-1})'q,\ q) - B_2B_1^{-1}\xi^1(-(B_2B_1^{-1})'q,\ q)$$

or

$$z^C(q) = \xi^2(-(C_2C_1^{-1})'q,\ q) - C_2C_1^{-1}\xi^1(-(C_2C_1^{-1})'q,\ q)\ .$$

(Of course, $p = -(C_2C_1^{-1})'q = -(B_2B_1^{-1})'q$.) In fact, for our purposes, $z(q)$ can be any convex combination of the two, $tz^B(q) + (1-t)z^C(q)$,

$0 \leq t \leq 1$.

Assumption 6 implies that there are only a finite number of prices $q \in S$ where $z(q)$ has multiple values. Consequently the homogeneity assumption implies that $z(q)$ is a differentiable function for almost all prices $q \in R^2_+$. It is worth pointing out, however, that for $z(q)$ to be multi-valued at an equilibrium q is in no way a degenerate situation. Although in the case, the linear programming problem is degenerate, the economy itself nee not be. It is quite possible for (ξ,A) to be a regular economy and to have n-1 activities in use at equilibrium $(\hat{p},\hat{q})$. If so, no small perturbation in (ξ,A) can result in fewer than n-1 activities being used at the perturbed equilibrium.

Let us investigate the situation where $\hat{q}$ is an equilibrium of $(z, -I)$ and there are two distinct bases $\begin{bmatrix} -\hat{q}'B_2 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} -\hat{q}'C_2 \\ C_1 \end{bmatrix}$ possible in the solution to our linear programming problem. There is some neighborhood of $\hat{q}$, $U \subset R^2$, such that the programming problem always results in either the basis $\begin{bmatrix} -\hat{q}'B_2 \\ B_1 \end{bmatrix}$ or $\begin{bmatrix} -\hat{q}'C_2 \\ C_1 \end{bmatrix}$. Let $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ be the activities associated with all those $q \in U \cap S$ such that $q_1 < \hat{q}_1$, and let $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ be the activities associated with those $q \in U \cap S$ such that $q_1 > \hat{q}_1$. We can choose some $v \in R^2$ with $v_1 = -v_2 > 0$ so that both $\hat{q}+v$ and $\hat{q}-v$ are elements of $U \cap S$. That $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is not associated with the basis at $\hat{q} - v$ implies that

$$-(\hat{q}-v)'B_2B_1^{-1}C_1 + (\hat{q}-v)'C_2 \leq 0$$

with one inequality holding strictly. Since C_1^{-1} has all elements

strictly positive,

$$-(\hat{q}-v)'B_2B_1^{-1} + (\hat{q}-v)'C_2C_1^{-1} < 0 .$$

Note, however, that $\hat{q}'B_2B_1^{-1} = \hat{q}'C_2C_1^{-1} = \hat{p}$. Consequently,

$$v'B_2B_1^{-1} < v'C_2C_1^{-1} .$$

Hence

$$v'B_2B_1^{-1}\xi^1(\hat{p},\hat{q}) < v'C_2C_1^{-1}\xi^1(\hat{p},\hat{q})$$

because we have assumed that $\xi^1(\hat{p},\hat{q})$ is strictly positive. Therefore,

$$v'\xi^2(\hat{p},\hat{q}) - v'B_2B_1^{-1}\xi^1(\hat{p},\hat{q}) > v'\xi^2(\hat{p},\hat{q}) - v'C_2C_1^{-1}\xi^1(\hat{p},\hat{q})$$

$$v'z^B(\hat{q}) > v'z^C(\hat{q}) .$$

Thus $v_1 > 0$ implies that $z_1^B(\hat{q}) > z_1^C(\hat{q})$, and $v_2 < 0$ implies that $z_2^B(\hat{q}) < z_2^C(\hat{q})$. It follows that whenever the excess demand correspondence is multi-valued at an equilibrium $\hat{q}$ the correspondence $z_1(q, 1-q_1)$ crosses the axis from above at $\hat{q}$ as in Figure 3.

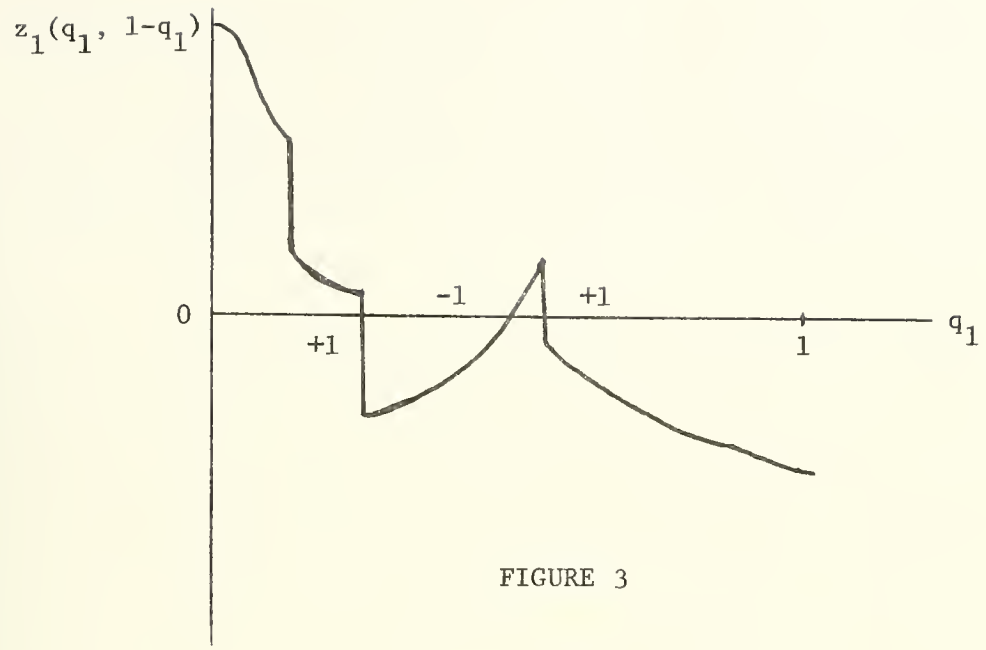

FIGURE 3

If $z(q)$ is not a differentiable function, we can no longer employ the index theorem. Nonetheless, if we assign $(\hat{q}) = +1$ to an equilibrium $\hat{q}$ whenever $z(\hat{q})$ is multi-valued and otherwise define index as before, we can use the intermediate value theorem of calculus and a simple counting argument to demonstrate that Σ index $(q) = +1$ when summed over all equilibria. It should come as no surprise that we set index $(\hat{q})$ equal to +1 if $z(\hat{q})$ is multi-valued. In this case $\hat{q}$ corresponds to an equilibrium $(\hat{p},\hat{q})$ of the economy (ξ, A) where there are n-1 activities in use. Hence index $(\hat{p},\hat{q}) = +1$.

The above arguments illustrate how an input-output model with two factors can be transformed into a two dimensional pure exchange model. An essential feature of this transformation is that the concept of index of equilibrium carries over. It also simplifies the problem of computing equilibria. This problem is reduced to a line search over the one dimensional price simplex $\{(q_1, q_2) | q_1 + q_2 = 1, q_j \geq 0\}$, which is easy to perform on a

computer. The advantage of this formulation is that it enables us to find all of the equilibria of an economy. By performing a line search of the one dimensional simplex, we can find all of the zeros of $z(q)$. In higher dimensions such an exhaustive search is seldom feasible.

8. CONCLUDING REMARKS

The conclusions of this paper are necessarily pessimistic. Although the index theorem provides us with necessary and sufficient mathematical conditions for uniqueness, we have been unable to use it to derive new conditions on the parameters of an economy that guarantee uniqueness and have meaningful economic interpretations. In fact, the new results that we have been able to derive are negative ones. Although it may yet be possible to combine conditions on both the consumption and production sides of an economy to guarantee uniqueness, it does not seem that such conditions can have very general applicability.

Previous researchers are to be admired for developing most of the significant economic theorems dealing with uniqueness without having such a powerful tool as the index theorem at their disposal. Indeed, as Mas-Colell once remarked to the writer, it would be surprising if there were many significant economic theorems to be derived after so many powerful minds have turned their attention to the question of uniqueness of equilibrium. Nonetheless, many of these researchers are to be faulted for two basic defects in approach. First, they have given too much attention to the pure exchange model, which misses many of the subtleties of the more general situation. Second, that previous approaches dealt with sufficient, rather than necessary, conditions has led many to optimistically believe that more general conditions could be derived. There are, of course, exceptions. The work of Rader, which was mentioned previously, provides a good example.

Nonetheless, non-uniqueness of equilibrium does not seem so pathological a situation as to warrant unqualified use of the simple comparative statics method when dealing with general equilibrium models. At present, many researchers are using variants of Scarf's algorithm to evaluate the implications of policy decisions in empirical general equilibrium models. In fact, the writer is one of them (see Kehoe and Serra [1980]). Yet, for most of these models no method now exists to determine whether the equilibrium found by the algorithm is unique. To make matters worse, it appears that non-uniqueness of equilibria is an even more common situation in applied models that allow for such distortions as taxes, price rigidities, and unemployment than it is in the simple model we have investigated here. There is a suggestion of this result in the work of Foster and Sonnenschein [10]; Kehoe [19] presents a demonstration using tools similar to those used in this paper. Some justification can be surely be given for working with an historically given equilibrium situation. Care must be taken, however, once the parameters of the economy shift in response to a policy change. If there is more than one equilibrium, then the problem becomes a dynamic one. If nothing else, a general set of conditions must be developed to ensure that the original equilibrium is locally stable under some realistic dynamic adjustment process. An even more complex problem arises as the parameters move through critical economies, where mathematical catastrophes can occur. Much work on these problems obviously remains to be done.

REFERENCES

[1] Arrow, K., H. Block, and L. Hurwicz: "On the Stability of the Competitive Equilibrium II," Econometrica, 27 (1959), 82-109.

[2] Arrow, K., and F. Hahn: General Competitive Analysis. San Francisco: Holden-Day, Inc., 1971.

[3] Chipman, J.: "Homothetic Preferences and Aggregation", Journal of Economic Theory, 8 (1974), 26-38.

[4] Debreu, G.: "Economies with a Finite Set of Equilibria," Econometrica, 38 (1970), 387-393.

[5] Debreu G.: "Smooth Preferences", Econometrica, 40 (1972), 603-612.

[6] Debreu, G. and I. Herstein: "Nonnegative Square Matrices", Econometrica, 21 (1953), 597-607.

[7] Dierker, E.: "Two Remarks on the Number of Equilibria of an Economy," Econometrica, 40, 951-953.

[8] Diewert, E.: "Applications of Duality Theory," in Frontiers of Quantitative Economics ed. by M. Intriligator and D. Kendrick. Volume 2. Amsterdam: North Holland, 1974, 106-171.

[9] Eaves, B. and H. Scarf: "The Solution of Systems of Piecewise Linear Equations," Mathematics of Operations Research, 1 (1976) 1-27.

[10] Foster, E. and H. Sonnenschein: "Price Distortions and Economic Welfare." Econometrica, 38 (1970), 281-297.

[11] Gale, D.:"A Note on Revealed Preference", Econometrica, 27 (1960), 348-354.

[12] _____ :The Theory of Linear Economic Models. New York: McGraw-Hill Book Company, 1960.

[13] Gale, D., and H. Nikaido: "The Jacobian Matrix and Global Univalence of Mappings." Mathematische Annalem, 159 (1965), 81-93.

[14] Gantmacher, F.: The Theory of Matrices. Volume I. Translated from the Russian by K. Hirsch. New York: Chelsea Publishing Company, 1959.

[15] Gorman, W.: "Community Preference Fields," Econometrica, 21 (1953) 63-80.

[16] Kehoe, T.: "Regularity and Index Theorem for Economic Equilibrium Models", Ph.D. Dissertation, Yale University, 1979.

[17] _____ : "Regular Production Economics," mimeo, Wesleyan University, 1979.

[18] _____ : "An Index Theorem for General Equilibrium Models with Production", Econometrica, 48 (1980), 1211-1232.

[19] _____ : "An Index Theorem for Tax Models", mimeo, M.I.T., 1980.

[20] _____ : "Index Theorems and Uniqueness of Equilibrium", mimeo, M.I.T., 1980.

[21] Kehoe, T. and J. Serra: "A General Equilibrium Model with Endogenous Unemployment", mimeo, M.I.T., 1980.

[22] Kihlstrom R., A Mas-Colell, and H. Sonnenschein: "The Demand Theory of the Weak Axiom of Revealed Preference", Econometrica, 44 (1976) 971-978.

[24] McKenzie, L.: "On the Existence of General Equilibrium for a Competitive Market," Econometrica, 27 (1959), 54-71.

[25] _____ : "On the Existence of General Equilibrium for a Competitive Market: Some Corrections," Econometrica, 29 (1961), 247-248.

[26] Mas-Colell, A. : "On the Continuity of Equilibrium Prices in Constant-Returns Production Economies," Journal of Mathematical Economics, 2 (1975), 21-33.

[27] _____ : "The Theory of Economic Equilibria from the Differentiable Point of View", Course Notes, Department of Mathematics, University of California, Berkeley, 1976.

[28] _____ : "An Introduction to the Differentiable Approach to the Theory of Economic Equilibria", in Studies in Mathematic Economics, ed. by S. Reiter. Washington, D.C.: Mathematical Association of America, forthcoming.

[29] Rader, T.: "Normally Factor Imputs are Never Gross Substitutes", Journal of Political Economy, 76 (1968), 38-43.

[30] Saigal, R., and C. Simon: "Generic Properties of the Complementary Problem", Mathematical Programming, 4 (1973), 324-335.

[31] Samuelson, P.: "Abstract of a Theorem Concerning Substitutability in Open Leontief Models", in Activity Analysis of Production and Allocation ed. by T. Koopmans. New York: Wiley, 1951, 142-146.

[32] Scarf, H. (in collaboration with T. Hansen): The Computation of Economic Equilibria. New Haven, Connecticut: Yale University Press, 1973.

[33] Todd, M.: "A Note on Computing Equilibria in Economies with Activity Analysis Models of Production", Journal of Mathematical Economics, 6 (1979), 135-144.

[34] Varian, H.: "A Third Remark on the Number of Equilibria of an Economy", Econometrica, 43(1974), 985-986.

[35] _____ : Microeconomic Analysis. New York: W.W. Norton, 1978.

[36] Wald, A.: "On Some Systems of Equations in Mathematical Economics," Econometrica, 19 (1951), 368-403. English translation of "Über Einige Gleichungssysteme der Mathematischem Ökonomie," Zeitschrift für Nationalokonomie, 7 (1936), 637-670.

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